

## ON KILLERS OF CABLE KNOT GROUPS

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ABSTRACT. A killer of a group  $G$  is an element that normally generates  $G$ . We show that the group of a cable knot contains infinitely many killers such that no two lie in the same automorphic orbit.

## 1. INTRODUCTION

Let  $G$  be an arbitrary group and  $S \subseteq G$ . We define the normal closure  $\langle\langle S \rangle\rangle_G$  of  $S$  as the smallest normal subgroup of  $G$  containing  $S$ , equivalently

$$\langle\langle S \rangle\rangle_G = \left\{ \prod_{i=1}^k u_i s_i^{\varepsilon_i} u_i^{-1} \mid u_i \in G, \varepsilon_i = \pm 1, s_i \in S, k \in \mathbb{N} \right\}.$$

Following [5], we call an element  $g \in G$  a *killer* if  $\langle\langle g \rangle\rangle_G = G$ . We say that two killers  $g_1, g_2 \in G$  are *equivalent* if there exists an automorphism  $\phi : G \rightarrow G$  such that  $\phi(g_1) = g_2$ .

Let  $\mathfrak{k}$  be a knot in  $S^3$  and  $V(\mathfrak{k})$  a regular neighborhood of  $\mathfrak{k}$ . Denote by

$$X(\mathfrak{k}) = S^3 - \text{Int}(V(\mathfrak{k}))$$

the knot manifold of  $\mathfrak{k}$  and by  $G(\mathfrak{k}) = \pi_1(X(\mathfrak{k}))$  its group. A *meridian* of  $\mathfrak{k}$  is an element of  $G(\mathfrak{k})$  which can be represented by a simple closed curve on  $\partial V(\mathfrak{k})$  that is contractible in  $V(\mathfrak{k})$  but not contractible in  $\partial V(\mathfrak{k})$ . Thus a meridian is well defined up to conjugacy and inversion.

From a Wirtinger presentation of  $G(\mathfrak{k})$  we see that the meridian is a killer. In [6, Theorem 3.11] the author exhibit a knot for which there exists a killer that is not equivalent to the meridian. Silver–Whitten–Williams [4, Corollary 1.3] showed that if  $\mathfrak{k}$  is a hyperbolic 2-bridge knot or a torus knot or a hyperbolic knot with unknotting number one, then its group contains infinitely many pairwise inequivalent killers.

In [4, Conjecture 3.3] it is conjectured that the group of any nontrivial knot has infinitely many inequivalent killers, see also [1, Question 9.26]. In this paper we show the following.

**Theorem 1.** *Let  $\mathfrak{k}$  be a cable knot about a nontrivial knot  $\mathfrak{k}_1$ . Then its group contains infinitely many pairwise inequivalent killers.*

Moreover, we show that having infinitely many inequivalent killers is preserved under connected sums. As a Corollary we show that the group of any nontrivial knot whose exterior is a graph manifold contains infinitely many inequivalent killers.

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## 2. PROOF OF THEOREM 1

Let  $m, n$  be coprime integers with  $n \geq 2$ . The *cable space*  $CS(m, n)$  is defined as follows: let  $D^2 = \{z \in \mathbb{C} \mid \|z\| \leq 1\}$  and  $\rho : D^2 \rightarrow D^2$  a rotation through an angle of  $2\pi(m/n)$  about the origin. Choose a disk  $\delta \subset \text{Int}(D^2)$  such that  $\rho^i(\delta) \cap \rho^j(\delta) = \emptyset$  for  $1 \leq i \neq j \leq n$  and denote by  $D_n^2$  the space

$$D^2 - \text{Int}\left(\bigcup_{i=1}^n \rho^i(\delta)\right).$$

$\rho$  induces a homeomorphism  $\rho_0 := \rho|_{D_n^2} : D_n^2 \rightarrow D_n^2$ . We define  $CS(m, n)$  as the mapping torus of  $\rho_0$ , i.e.

$$CS(m, n) := D_n^2 \times I / (z, 0) \sim (\rho_0(z), 1).$$

Note that  $CS(m, n)$  has the structure of a Seifert fibered space. Each fiber is the image of  $\{\rho^i(z) \mid 1 \leq i \leq n\} \times I$  under the quotient map, where  $z \in D_n^2$ . There is exactly one exceptional fiber, namely the image  $C_0$  of the arc  $0 \times I$ .

In order to compute the fundamental group  $A$  of  $CS(m, n)$ , denote the free generators of  $\pi_1(D_n^2)$  corresponding to the boundary paths of the removed disks  $\rho_0(\delta), \dots, \rho_0^n(\delta)$  by  $x_1, \dots, x_n$  respectively. From the definition of  $CS(m, n)$  we see that we can write  $A$  as the semi-direct product  $F(x_1, \dots, x_n) \rtimes \mathbb{Z}$ , where the action of  $\mathbb{Z} = \langle t \rangle$  on  $\pi_1(D_n^2) = F(x_1, \dots, x_n)$  is given by

$$tx_it^{-1} = x_{\sigma(i)} \text{ for } 1 \leq i \leq n.$$

The element  $t$  is represented by the exceptional fiber of  $CS(m, n)$  and the permutation  $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$  is given by  $i \mapsto i + m \bmod n$ . Thus

$$A = \langle x_1, \dots, x_n, t \mid tx_it^{-1} = x_{\sigma(i)} \text{ for } 1 \leq i \leq n \rangle.$$

We finally remark that any element  $a \in A$  is uniquely written as  $w \cdot t^z$  for some  $w \in F(x_1, \dots, x_n)$  and  $z \in \mathbb{Z}$ .

We next define cable knots. Let  $V_0$  be the solid torus  $D^2 \times I / (z, 0) \sim (\rho(z), 1)$  and for some  $z_0 \in \text{Int}(D^2) - 0$  let  $\mathfrak{k}_0$  be the image of  $\{\rho^i(z_0) \mid 1 \leq i \leq n\} \times I$  under the quotient map. Note that  $\mathfrak{k}_0$  is a simple closed curve contained in the interior of  $V_0$ . Let  $\mathfrak{k}_1$  be a nontrivial knot in  $S^3$  and  $V(\mathfrak{k}_1)$  a regular neighborhood of  $\mathfrak{k}_1$  in  $S^3$ . Let further  $h : V_0 \rightarrow V(\mathfrak{k}_1)$  be a homeomorphism which maps the meridian  $\partial D^2 \times 1$  of  $V_0$  to a meridian of  $\mathfrak{k}_1$ . The knot  $\mathfrak{k} := h(\mathfrak{k}_0)$  is called a  $(m, n)$ -cable knot about  $\mathfrak{k}_1$ .

Thus the knot manifold  $X(\mathfrak{k})$  of a  $(m, n)$ -cable knot  $\mathfrak{k}$  decomposes as

$$X(\mathfrak{k}) = CS(m, n) \cup X(\mathfrak{k}_1)$$

with  $\partial X(\mathfrak{k}_1) = CS(m, n) \cap X(\mathfrak{k}_1)$  an incompressible torus in  $X(\mathfrak{k})$ . It follows from the Theorem of Seifert and van-Kampen that

$$G(\mathfrak{k}) = A *_C B$$

where  $B = G(\mathfrak{k}_1)$  and  $C = \pi_1(\partial X(\mathfrak{k}_1))$ . Denote by  $m_1$  the meridian of  $\mathfrak{k}_1$  and note that in  $A$  we have  $m_1 = x_1 \cdot \dots \cdot x_n$ . In turn, the meridian  $m \in G(\mathfrak{k})$  of  $\mathfrak{k}$  is written as  $m = x_1 \in A$ .

The proof of Theorem 1 is divided in two steps. In Lemma 1 we exhibit elements that normally generate the group of the cable knot and next, in Lemma 2, we prove that these killers are indeed inequivalent.

Choose  $s \in \{1, \dots, n-1\}$  such that  $\sigma^s(1) = 2$ . Since  $\sigma^s(i) = i + sm \pmod n$  it follows that  $\sigma^s = (1 \ 2 \ 3 \ \dots \ n-1 \ n)$ .

**Lemma 1.** *Let  $\mathfrak{k}$  be a  $(m, n)$ -cable knot about a nontrivial knot  $\mathfrak{k}_1$ . Then for each  $l \geq 1$  the element*

$$g_l := x_1^l x_2^{-(l-1)} = x_1^l \cdot (t^s x_1 t^{-s})^{-(l-1)}$$

*normally generates the group of  $\mathfrak{k}$ .*

*Proof.* The first step of the proof is to show that the group of the companion knot is contained in  $\langle\langle g_l \rangle\rangle_{G(\mathfrak{k})}$ .

**Claim 1:** The meridian  $m_1 = x_1 \cdot \dots \cdot x_n$  of  $\mathfrak{k}_1$  belongs to  $\langle\langle g_l \rangle\rangle_{G(\mathfrak{k})}$ . Consequently  $B = \langle\langle m_1 \rangle\rangle_B = \langle\langle m_1 \rangle\rangle_{G(\mathfrak{k})} \cap B \subseteq \langle\langle g_l \rangle\rangle_{G(\mathfrak{k})}$ .

Note that for  $0 \leq i \leq n-1$  we have  $t^{is} g_l t^{-is} = x_{i+1}^l x_{i+2}^{-(l-1)}$ , where indices are taken mod  $n$ . Thus

$$\begin{aligned} x_1 \cdot \dots \cdot x_n &= x_1^{-(l-1)} (x_1^l x_2 \cdot \dots \cdot x_n x_1^{-(l-1)}) x_1^{l-1} \\ &= x_1^{-(l-1)} \left( \prod_{i=0}^{n-1} x_{i+1}^l x_{i+2}^{-(l-1)} \right) x_1^{l-1} \\ &= x_1^{-(l-1)} \left( \prod_{i=0}^{n-1} t^{is} \cdot g_l \cdot t^{-is} \right) x_1^{l-1} \\ &= \prod_{i=0}^{n-1} \left( x_n^{-(l-1)} t^{is} \cdot g_l \cdot t^{-is} x_1^{l-1} \right) \\ &= \prod_{i=0}^{n-1} \left( (x_1^{-(l-1)} t^{is}) \cdot g_l \cdot (x_1^{-(l-1)} t^{is})^{-1} \right) \end{aligned}$$

which implies that  $m_1 \in \langle\langle g_l \rangle\rangle_{G(\mathfrak{k})}$ . Thus Claim 1 is proved.

From Claim 1 it follows that the peripheral subgroup  $C = \pi_1(\partial X(\mathfrak{k}_1))$  of  $\mathfrak{k}_1$  is contained in  $\langle\langle g_l \rangle\rangle_{G(\mathfrak{k})}$  since  $C \subseteq B$  and consequently we have

$$G(\mathfrak{k}) / \langle\langle g_l \rangle\rangle_{G(\mathfrak{k})} = A *_C B / \langle\langle g_l \rangle\rangle_{G(\mathfrak{k})} \cong A / \langle\langle g_l, C \rangle\rangle_A.$$

Thus, we need to show that  $A / \langle\langle g_l, C \rangle\rangle_A = 1$ . It is easy to see that  $A / \langle\langle C \rangle\rangle_A$  is cyclically generated by  $\pi(x_1)$ , where  $\pi : A \rightarrow A / \langle\langle C \rangle\rangle_A$  is the canonical projection. The result now follows from the fact that  $\pi(g_l) = \pi(x_1^l \cdot t^s x_1^{-(l-1)} t^{-s}) = \pi(x_1)$ .  $\square$

**Lemma 2.** *If  $k \neq l$ , then  $g_k$  is not equivalent to  $g_l$ .*

*Proof.* Assume that  $\phi : G(\mathfrak{k}) \rightarrow G(\mathfrak{k})$  is an automorphism such that  $\phi(g_l) = g_k$  and let  $f : X(\mathfrak{k}) \rightarrow X(\mathfrak{k})$  be a homotopy equivalence inducing  $\phi$ . From [3, Theorem 14.6] it follows that  $f$  can be deformed into  $\hat{f} : X(\mathfrak{k}) \rightarrow X(\mathfrak{k})$  so that  $\hat{f}$  sends  $X(\mathfrak{k}_1)$  homeomorphically onto  $X(\mathfrak{k}_1)$  and  $\hat{f}|_{CS(m,n)} : CS(m,n) \rightarrow CS(m,n)$  is a homotopy equivalence. Thus  $\phi(A)$  is conjugated to  $A$ , that is,  $\phi(A) = gAg^{-1}$  for some  $g \in G(\mathfrak{k})$ . Since  $\phi(g_l) = g_k$  it implies that  $g_k \in gAg^{-1}$ . As  $g_k$  is not conjugated (in  $A$ ) to an element of  $C$ , it implies that  $g \in A$  and so  $\phi(A) = A$ . By [3, Proposition 28.4], we may assume that  $\hat{f}|_{CS(m,n)}$  is fiber preserving. Since  $CS(m,n)$  has exactly one exceptional fiber, which represents  $t$ , we must have  $\phi(t) = at^\eta a^{-1}$  for some  $a = v \cdot t^{z_1} \in A$  and some  $\eta \in \{\pm 1\}$ .

The automorphism  $\phi|_A : A \rightarrow A$  induces an automorphism  $\phi_*$  on the factor group  $A / \langle t^n \rangle = \langle x_1, t \mid t^n = 1 \rangle = \mathbb{Z} * \mathbb{Z}_n$  such that  $\phi_*(t) = at^\eta a^{-1}$ . It is a standard fact

about automorphisms of free products that we must have  $\phi_*(x_1) = at^{e_0}x_1^\varepsilon t^{e_1}a^{-1}$  for  $e_0, e_1 \in \mathbb{Z}$  and  $\varepsilon \in \{\pm 1\}$ . Thus,

$$\phi(x_1) = at^{e_0}x_1^\varepsilon t^{e_1}a^{-1}t^{dn} = at^{e_0} \cdot x_1 t^{e_0+e_1+dn} \cdot t^{-e_0}a^{-1}$$

for some  $d \in \mathbb{Z}$ . Since  $t$  has non-zero homology in  $H_1(X(\mathfrak{k}))$  it follows that  $e_0 + e_1 + dn = 0$ . Consequently,  $\phi(x_1) = b \cdot x_1^\varepsilon \cdot b^{-1}$ , where  $b = at^{e_0} = v \cdot t^{z_2} \in A$  and  $z_2 = z_1 + e_0$ .

Hence we obtain

$$\begin{aligned} \phi(g_l) &= \phi(x_1^l x_2^{-(l-1)}) \\ &= \phi(x_1^l \cdot t^s x_1^{-(l-1)} t^{-s}) \\ &= b x_1^{\varepsilon l} b^{-1} \cdot at^{\eta s} a^{-1} \cdot b x_1^{-\varepsilon(l-1)} b^{-1} \cdot at^{-\eta s} a^{-1} \\ &= v t^{z_2} x_1^{\varepsilon l} t^{-z_2} v^{-1} \cdot v t^{z_1} t^{\eta s} t^{-z_1} v^{-1} \cdot v t^{z_2} x_1^{-\varepsilon(l-1)} t^{-z_2} v^{-1} \cdot v t^{z_1} t^{-\eta s} t^{-z_1} v^{-1} \\ &= v x_i^{\varepsilon l} x_j^{-\varepsilon(l-1)} v^{-1} \end{aligned}$$

where  $i = \sigma^{z_2}(1)$  and  $j = \sigma^{z_2+\eta s}(1)$ . Note that  $i \neq j$  since  $\sigma^s(1) = 2$  and  $\sigma^{-s}(1) = n$ . Hence,  $\phi(g_l) = g_k$  implies that

$$v(x_i^{\varepsilon l} \cdot x_j^{-\varepsilon(l-1)})v^{-1} = x_1^k \cdot x_2^{-(k-1)}$$

in  $F(x_1, \dots, x_n)$ . Thus, in the abelinization of  $F(x_1, \dots, x_n)$  we have

$$\varepsilon[lx_i + (1-l)x_j] = kx_1 + (1-k)x_2$$

which implies that  $\{i, j\} = \{1, 2\}$ . If  $(i, j) = (1, 2)$ , then  $\varepsilon l = k$  and so  $k = |k| = |\varepsilon l| = l$ . If  $(i, j) = (2, 1)$ , then  $\varepsilon l = k - 1$  and  $\varepsilon(1 - l) = k$ . Consequently,  $\varepsilon = 1$  and  $l + k = 1$  which is impossible since  $k, l \geq 1$ .  $\square$

### 3. CONNECTED SUMS AND KILLERS

In this section we show that having infinitely many inequivalent killers is preserved under connected sums of knots. This fact, Theorem 1, and Corollary 1.3 of [4] imply that the group of knots whose exterior is a graph manifold have infinitely many inequivalent killers.

Let  $\mathfrak{k}$  be a knot and  $\mathfrak{k}_1, \dots, \mathfrak{k}_n$  its prime factors, that is,  $\mathfrak{k} = \mathfrak{k}_1 \# \dots \# \mathfrak{k}_n$  and each  $\mathfrak{k}_i$  is a nontrivial prime knot. Assume that  $x \in G(\mathfrak{k}_i)$  is a killer of  $G(\mathfrak{k}_i)$ . It is well-known that  $G(\mathfrak{k}_i) \leq G(\mathfrak{k})$  and  $\langle m \rangle \leq G(\mathfrak{k}_i)$  for all  $i$ , where  $m$  denotes the meridian of  $\mathfrak{k}$ . From this we immediately see that  $m \in \langle \langle x \rangle \rangle_{G(\mathfrak{k}_i)} \subseteq \langle \langle x \rangle \rangle_{G(\mathfrak{k})}$  which implies that  $G(\mathfrak{k}) = \langle \langle m \rangle \rangle_{G(\mathfrak{k})} \subseteq \langle \langle x \rangle \rangle_{G(\mathfrak{k})}$ , ie.,  $x$  is a killer of  $G(\mathfrak{k})$ .

Now suppose that  $x, y \in G(\mathfrak{k}_i)$  are killers of  $G(\mathfrak{k}_i)$  and that there exists an automorphism  $\phi$  of  $G(\mathfrak{k})$  such that  $\phi(x) = y$ .  $\phi$  is induced by a homotopy equivalence  $f : X(\mathfrak{k}) \rightarrow X(\mathfrak{k})$ . From [3, Theorem 14.6] it follows that  $f$  can be deformed into  $\hat{f} : X(\mathfrak{k}) \rightarrow X(\mathfrak{k})$  so that:

1.  $\hat{f}|_V : V \rightarrow V$  is a homotopy equivalence, where  $V = S^1 \times (n\text{-punctured disk})$  is the peripheral component of the characteristic submanifold of  $X(\mathfrak{k})$ .
2.  $\hat{f}|_{\overline{X(\mathfrak{k}) - V}} : \overline{X(\mathfrak{k}) - V} \rightarrow \overline{X(\mathfrak{k}) - V}$  is a homeomorphism.

Note that  $\overline{X(\mathfrak{k}) - V} = X(\mathfrak{k}_1) \dot{\cup} \dots \dot{\cup} X(\mathfrak{k}_n)$ . Since  $\hat{f}|_{\overline{X(\mathfrak{k}) - V}}$  is a homeomorphism it follows that  $\hat{f}$  sends  $X(\mathfrak{k}_i)$  homeomorphically onto  $X(\mathfrak{k}_{\tau(i)})$  for some permutation  $\tau$  of  $\{1, \dots, n\}$ . Consequently, there exists  $g' \in G(\mathfrak{k})$  such that

$$\phi(G(\mathfrak{k}_i)) = g'G(\mathfrak{k}_{\tau(i)})g'^{-1}.$$

If  $\tau(i) = i$  and  $g' \in G(\mathfrak{k}_i)$ , then  $\phi$  induces an automorphism  $\psi := \phi|_{G(\mathfrak{k}_i)}$  of  $G(\mathfrak{k}_i)$  such that  $\psi(x) = y$ , i.e.,  $x$  and  $y$  are equivalent in  $G(\mathfrak{k}_i)$ . If  $\tau(i) \neq i$  or  $g' \notin G(\mathfrak{k}_{\tau(i)})$ , then it is not hard to see that  $y$  is conjugated (in  $G(\mathfrak{k}_i)$ ) to an element of  $\langle m \rangle$  since  $y = \phi(x) \in G(\mathfrak{k}_i) \cap g'G(\mathfrak{k}_{\tau(i)})g'^{-1}$ . As  $\langle\langle m^k \rangle\rangle \neq G(\mathfrak{k})$  for  $|k| \geq 2$  and  $y$  normally generates  $G(\mathfrak{k})$ , we conclude that  $y$  is conjugated (in  $G(\mathfrak{k}_i)$ ) to  $m^{\pm 1}$ . The same argument applied to  $\phi^{-1}$  shows that  $x$  is conjugated (in  $G(\mathfrak{k}_i)$ ) to  $m^{\pm 1}$ .

Therefore, if the group of one of the prime factors of  $\mathfrak{k}$  has infinitely many inequivalent killers, then so does the group of  $\mathfrak{k}$ . As a Corollary of Theorem 1 and the remark made above we obtain the following result.

**Corollary 1.** *If  $\mathfrak{k}$  is a knot such that  $X(\mathfrak{k})$  is a graph manifold, then  $G(\mathfrak{k})$  contains infinitely many pairwise inequivalent killers.*

*Proof.* From [2] it follows that the only Seifert-fibered manifolds that can be embedded into a knot manifold with incompressible boundary are torus knot complements, composing spaces and cable spaces. Thus, if  $X(\mathfrak{k})$  is a graph manifold, then one of the following holds:

- (1)  $\mathfrak{k}$  is a torus knot.
- (2)  $\mathfrak{k}$  is a cable knot.
- (3)  $\mathfrak{k} = \mathfrak{k}_1 \# \dots \# \mathfrak{k}_n$  where each  $\mathfrak{k}_i$  is either a torus knot or a cable knot.

Now the result follows from Theorem 1 and [4, Corollary 1.3].  $\square$

#### REFERENCES

- [1] M. Aschenbrenner, S. Friedl, and H. Wilton, 3-manifold groups, arXiv:1205.0202v3 [math.GT].
- [2] W. Jaco and P. Shalen, *Seifert fibered spaces in 3-manifolds*, Mem. AMS **220**(1979).
- [3] K. Johanson, *Homotopy Equivalences of 3-manifolds with boundary*, Lectures Notes in Math., Vol. 761, Springer-Verlag, Berlin and New York, 1979.
- [4] D. S. Silver, W. Whitten, S. G. Williams, *Knot groups with many killers*, Bull. Aust. Math. Soc. **81** (2010), 507-513.
- [5] J. Simon, *Wirtinger approximations and the knot groups of  $F^n$  in  $S^{n+1}$* , Pacific J. Math **90** (1990), 177-189.
- [6] C. M. Tsau, *Nonalgebraic killers of knot groups*, Proc. Amer. Math. Soc. **95** (1985), 139-146.

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